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PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS' USE OF PARAMETERS IN EXPLAINING GEOMETRIC TRANSFORMATIONS[1]

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Abstract

National and international curriculum documents give emphasis to geometric transformations. In particular, parameters of geometric transformations play crucial role in defining geometric transformations. Thus, it needs to be explored to what extent prospective teachers use parameters in explaining geometric transformations. In this study, qualitative case study methodology was used and participants were sixteen prospective middle school mathematics teachers enrolled in a teacher education program in Turkey. To reveal prospective teachers', use of parameters in explaining geometric transformations, a transformation geometry activity was designed and implemented. During the implementation of the activity, classroom observations were carried out. The classroom observations focused on participants' small group discussions and whole class discussions regarding geometric transformation concepts. Moreover, implementation of the activity was audio-taped and video-taped. Participants' use of parameters was revealed by using written materials and participants' explanations during the whole class discussions. The findings showed that none of the pairs provided totally correct explanations for geometric transformations. They had partial understanding about the parameters of rotation. Moreover, one of the pairs held incorrect ideas about identification of the angle of rotation.

Keywords: Parameters of geometric transformations; prospective teachers; reflection line; angle of rotation; center of rotation

INTRODUCTION

Geometry is defined as "complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imagined spatial environments" (Battista, 2007, p. 843). Geometry assists students in mathematical reasoning, making and approving assumptions, and defining and classifying geometrical objects. Geometric ideas are important in representing and solving real life problems and in other mathematical areas (NCTM, 2000).

Van de Walle, Karp, and Bay-Williams (2016) defined spatial reasoning as an intuition about shapes and the relationships among them. They added that spatial reasoning includes the ability to mentally imagine objects and that it requires the ability to turn things in the mind. Spatial abilities can be regarded as a kind of cognitive activity that helps learners produce spatial images and use them in the solution of different problems (Hegarty & Waller, 2005). Meanwhile, studying transformation geometry

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concepts has the potential to foster students' reasoning and justification skills. Moreover, it gives learners opportunities to describe patterns, make generalizations, and develop spatial abilities (Portnoy, Grundmeier, & Graham, 2006). There is much in common between spatial reasoning and transformation geometry since both include construction and manipulation of mental images (Leong & Lim-Teo, 2003). Therefore, working with transformation geometry would improve skills in spatial and geometric reasoning (Clements & Battista, 1992; Edwards, 1997).

In particular, parameters play a crucial role in understanding geometric transformations. Actually, they are parts of the definitions of geometric transformations. Definition of reflection implies that a reflection is defined by a line and this line is called a reflection line (i.e., the parameter of a reflection). Reflection line determines the particular member of reflection family. Meanwhile, definition of a rotation implies that the parameters of a rotation are a point and an angle which are called the center of rotation and the angle of rotation. Angle of rotation and center of rotation together determine the particular member of rotation family. Translation vector determine the particular member of translation family. Thus, parameters are a line of reflection for a reflection, a vector for a translation, and a center of rotation and an angle of rotation for a rotation.

Although geometric transformations and their parameters are important in improving spatial and geometric reasoning skills (Clements & Battista, 1992; Edwards, 1997), research has shown that learners have various difficulties in understanding geometric transformations such as defining and identifying geometric transformations with their all parameters (e.g., Hollebrands, 2004; Thaqi et al., 2011; Yanık, 2011, 2014) and using appropriate mathematical terminology in describing geometric transformations (e.g., Harper, 2003). For instance, Hollebrands (2004) conducted a study to gain insights into high school students' understanding of geometric transformations. When students were asked to define a rotation, most of them explained it as "turning an object". All of them explicitly included the angle of rotation in their explanations while the center of rotation seemed to be implicit. In another study, Yanık (2011) explored prospective middle school mathematics teachers' preconceptions of geometric translations. In his subsequent study, Yanık (2014) examined middle-school students' concept images of geometric translations. The results of these two studies showed that neither middle school students nor prospective middle school teachers mentioned the magnitude or the direction of the translation vector when defining translation. Similarly, Thaqi et al. (2011) conducted semi-structured interviews with prospective elementary teachers to examine participants' use of parameters and their ideas about transformations. They found out that prospective elementary teachers mentioned only the angle of rotation but not the center of rotation when defining rotation.

Harper (2003) investigated pre-service elementary teachers' use of mathematical language and terminology about geometric transformations both before and after the implementation of instructional sessions through the use of dynamic geometry computer software, The Geometers' Sketchpad. She found out that the participants used informal mathematical terminology such as "flip, flop, fold, shift, move, slide, turn, spin, swing, and pivot" (p. 2912) prior to the instruction. However, participants' vocabulary of geometric transformations became more sophisticated and their use of appropriate mathematical terminology developed after the instruction. Specifically, the participants began to use formal terms such as "translation, translation vector, reflection, line of reflection, rotation, equidistance, perpendicularity, center of rotation, and angle of rotation".

In Turkey, K-12 mathematics curriculum put considerable emphasis on geometric transformation concepts. Transformation geometry has been integrated into middle school mathematics curriculum as a sub-learning domain of geometry after the implementation of reform movements in Turkey in 2005. Transformation geometry topics have maintained their importance thereafter. For instance, 8th grade mathematics curriculum includes the following three learning objectives about geometric transformations: (*i*) construct the images of translated points, line segments and other shapes, (*ii*) construct the images of reflected points, line segments and other shapes, and (*iii*) perform successive translations and reflections of polygons (Ministry of National Education, 2018). Given that all learning objectives regarding geometric transformations involve using parameters, it needs to be explored to what extent prospective teachers use parameters in explaining geometric transformations. Examining prospective teachers' use of parameters in geometric transformations is significant. Since they are going to teach geometric transformation concepts in the near future when they start teaching profession, they must have complete understanding of geometric transformations before entering into the classrooms. Therefore, to reveal prospective teachers' use of parameters in explaining geometric transformations, a transformation geometry activity was designed and implemented. Participants' use of parameters was revealed by focusing on their explanations during group discussions, whole class discussions and the written materials. Therefore, the research questions of this study were: (1) To what extent do prospective middle school mathematics teachers use reflection line in explaining reflection? and (2) To what extent do prospective middle school mathematics teachers use angle of rotation and center of rotation in explaining rotation?

METHODOLOGY

The aim of this study is to explore prospective teachers' use of parameters in explaining geometric transformations. In this part, the participants and the design of the study are explained and then the data collection procedures and data analysis procedure are explained.

Research Design and the Participants

In this study, qualitative case study methodology was used. The context of the study was middle school mathematics teacher education program. Senior and junior prospective mathematics teachers constituted the case of the study and the unit of analysis was their use of parameters in explaining geometric transformations. Participants of the study were sixteen volunteered prospective middle school mathematics teachers who were enrolled in a teacher education program at a state university in Central Anatolia. Convenience sampling method was used and there were 3 junior and 13 senior prospective teachers.

Data Collection

Qualitative research methods were used in data collection process. *Parameters in Finite Figures Activity* was developed and implemented in two iterations by the researcher. During the implementation of the activity, classroom observations were carried out by the researcher (i.e., the author of the study) and another researcher who had a doctoral degree in mathematics education (i.e., non-participant observer). The classroom observations focused on participants' small group discussions and whole class discussions regarding geometric transformation concepts. Moreover, implementation of the activity was audio-taped and video-taped. There were three video cameras in the classroom and audio recordings for each small group. One camera was placed at the back of the classroom. This camera recorded participants' verbal and written explanations for tasks on the smart board. The second and the third camera were placed in front of the classroom and focused on the classroom interactions that took place between the researcher and the participants and among participants. Finally, written materials completed by the participants were collected.

In the *Parameters in Finite Figures Activity,* participants were expected to explain finite figures by means of geometric transformations and with their all parameters. The finite figures presented to the participants in this activity included either only rotation or both rotation and reflection. All finite figures used in this activity are presented in Figure 1.

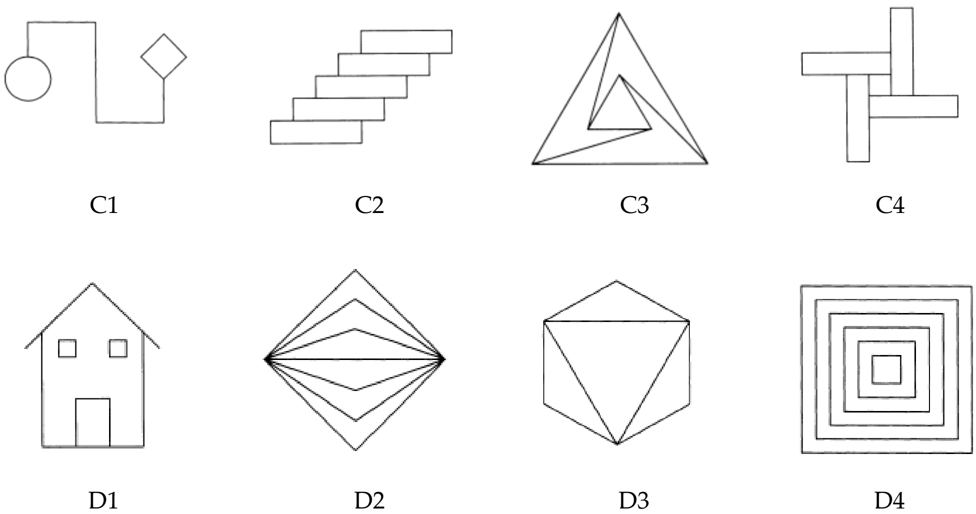


Figure 1. Finite Figures Included in the Activity (taken from Farmer, 1996, pp: 36-38).

To illustrate, characteristics of the finite figures C3 and D3 are compared as follows. While the third finite figure (C3) in Figure 1 involves only 120 degrees rotations, the seventh finite figure (D3) involves both 120 degrees rotation and three reflections. Therefore, the pair selecting one of the finite figures involving both reflection and rotation (D1, D2, D3, or D4) had to explain their figure by using both reflection and rotation with their parameters to differentiate D3 from C3 or vice versa.

The Parameters in Finite Figures Activity consisted of two parts and the participants were expected to work in pairs. In the first part, each pair randomly selected a finite figure from a bag of eight finite figures and they examined the figure without telling or showing it to other pairs. Here, the participants were expected to identify the geometric transformations used in the construction of the finite figures and explain them by means of parameters of geometric transformations. Each pair wrote their own explanation on a blank paper and put their papers to the empty bag. In the second part of the activity, the researcher mixed the bag and requested each pair to select one paper on which there was explanation of a finite figure written by another pair in the first part. Meanwhile, eight figures were projected onto the smart board. The pairs tried to match the written explanations in their hands with figures that were projected onto the smart board. After a while, all pairs read the written explanation in their papers to the class loudly. To let other pairs think enough on each written explanation, the reading pair did not immediately tell their answer to the class. The whole class discussion went on until all matches were completed correctly.

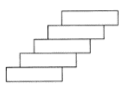
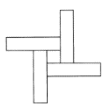
Data Analysis Procedures

The written materials and the audio recordings of pairs and whole class discussions were used to analyze participants' use of parameters in explaining geometric transformations with their parameters during the implementation of the activity. Video recordings of classroom session and audio recordings of pairs and whole class discussions were transcribed by the researcher. Since participants shared their responses with the other participants in the whole class discussions, the researcher tried to find clues related to participants' use of parameters in explaining geometric transformations with their parameters.

FINDINGS

In this part, findings related to prospective teachers' use of parameters to define reflection and rotation are presented. As explained in the methodology section, the participants were expected to explain finite figures by means of geometric transformations and with their parameters in the first part of the activity. In general, prospective teachers were presented 8 rotations (8 angle of rotations and 8 center of rotations) and 4 reflections (4 reflection lines) altogether. Findings showed that none of the eight pairs included the center of the rotations and half of them (4 pairs) included the angle of rotation in their explanations. Besides, one of the pairs did not include the reflection line in their explanations. In the following paragraphs, the evaluation of prospective teachers' written explanations is presented. First, findings related to the figures including only rotation were explained. Finite figures and participants' written explanations for these finite figures are presented in Table 1.

Table 1. The Finite Figures Including Only Rotation and the Corresponding Written Explanations

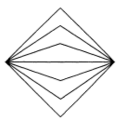
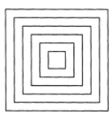
Randomly selected figure	Participants' explanations of the figure via geometric transformations and their parameters	Explanation number	Explained by
	No transformation was applied.	1	PT5 & PT6
	Rotation was applied. It was rotated 180 degrees.	2	PT9 & PT10
	Rotation was applied. It was obtained by 90 degrees rotations. 90 degrees rotation was applied twice.	3	PT7 & PT8
	Rotation was applied. It was obtained by applying 90, 180 and 270 degrees rotations to the smallest unit.	4	PT1 & PT2

All of the finite figures presented in Table 1 include rotation with different angle measures and the centers of the rotations are the centers of the figures themselves. However, none of them include reflection. Actually, they all have finite cyclic symmetries. Correctness of prospective teachers' written explanations is reported in the following paragraphs.

As seen in Table 1, the first figure has 360 degrees rotation (symmetry type C1). However, PT5 & PT6 expressed that "no transformation was applied" and they did not mention that 360 degrees rotation of the figure yielded itself. Thus, their explanation cannot be accepted as correct. The second figure has 180 degrees rotation (symmetry type C2). PT9 & PT10's explained the angle of rotation correctly but they did not mention the other parameter, namely the center of rotation.

The third figure in Table 1 has 120 degrees rotation (symmetry type C3). The 120 degrees rotations were applied three times around the center of the figure. PT7 & PT8 stated that rotation was applied. This was correct but they could not identify the angle of rotation correctly. Indeed, there is a contradiction in this explanation because if the figure has two rotation symmetries, the angle of rotation must be equal to 180 degrees. If the angle of rotation is 90 degrees as written in the explanation, then the figure must have four rotation symmetries. Thus, it can be said that PT7 & PT8 had some trouble about the angle of rotation. Besides, they did not mention the center of rotation in explaining their figure. The fourth figure (symmetry type C4) was explained by PT1 & PT2. They identified the angle of rotation correctly. Like the previous three pairs, they did not mention the center of rotation in their explanation and thus their explanation was accepted as partially correct. In the next part, the findings related to figures including both reflection and rotation are presented. These figures are presented in Table 2.

Table 2. The Finite Figures Including Both Reflection and Rotation and the Corresponding Written Explanations

Randomly selected figure	**Explanations of the figure via geometric transformations and their parameters**	**Explanation number**	**Explained by**
	Our figure can only be obtained by reflecting the unit. There is only one reflection axis.	5	PT15 & PT16
	There are only two reflection lines. These are *x*-axis and *y*-axis. Rotation was not applied.	6	PT13 & PT14
	This figure was obtained by rotating the unit three times.	7	PT3 & PT4
	Four reflection symmetries and four rotation symmetries	8	PT11 & PT12

Figures presented in Table 2 include both reflection and rotation. The number of rotation symmetries is equal to the number of reflection symmetries for each figure and thus they all have dihedral symmetry. Each figure has different angle of rotation and the centers of the rotations are the centers of the figures themselves. Correctness of prospective teachers' written explanations is presented below.

The fifth figure has 360 degrees rotation and it has one reflection line (symmetry type D1). Since PT15 & PT16 explained that the fifth figure includes only reflection, their answer was accepted as partially correct. PT13 & PT14 correctly stated that the sixth figure has two reflection lines. However, they emphasized that rotation was not applied. There was actually a contradiction in their explanation since in a figure which has reflection symmetry, there must absolutely be rotation symmetry (symmetry type D2). Thus, their explanation of the sixth figure was also accepted as partially correct.

PT3 & PT4 correctly explained that the seventh figure can be obtained by rotating the unit three times (symmetry type D3). However, they did not mention about three reflections in the figure so their explanation was accepted as partially correct. The eighth figure includes four reflections and four rotation symmetries (symmetry type D4). Although, PT11 & PT12 did not explicitly state the center of rotation, their explanation included all symmetry types and thus their explanation was regarded as almost correct.

To date, correctness of prospective teachers' written explanations for finite figures using geometric transformations were presented based on the written materials completed by pairs. To summarize, none of the pairs provided totally correct explanations for all eight figures. For the figures that only include rotation, three pairs identified rotation and two pairs wrote the angle of these rotations. However, none of the pairs included the center of rotation in their explanations. They had partial understanding about the parameters of rotation. Even, one of the pairs held incorrect ideas about identification of the angle of rotation.

For the figures that include both rotation and reflection, three pairs identified only one of these transformations, namely either rotation or reflection. Those who identified reflections could also identify their parameters, namely the reflection lines. One pair identified only rotation and its angle but could not identify its center. Finally, one pair identified both reflection and rotation and expressed these transformations with reflection lines and with an angle of rotation. None of the pairs, including the final one, mentioned the center of rotation when explaining their figures.

As mentioned before, pairs were first requested to explain the figures by means of geometric transformations. Then, in the second part of the activity, their written explanations were put into a bag. Finally, the pairs randomly selected the explanations written by others. Meanwhile, the eight figures were projected onto the smart board. The pairs tried to match the written explanations with figures that were projected onto the smart board. In the following paragraphs, prospective teachers' efforts to match the written explanations with the related figures are presented.

PT1 & PT2 selected the eighth explanation from the bag. They read loudly this explanation and they were able to decide on the eighth figure immediately and correctly. It is important to note that the eighth explanation written by PT11 & PT12 was almost correct. Therefore, PT1 & PT2 were able match the figure correctly.

PT15 & PT16 selected the sixth explanation written by PT13 & PT14 and read it loudly. However, they were not able to make any comment on the written explanation. The other participants were very silent as well. Despite being incorrect, PT13 & PT14 explicitly stated that "rotation was not applied" in their written explanation. One reason for this silence was the misleading incorrect information. After some time, PT11 stated that "only the sixth figure had two reflection lines". PT15 interrupted and claimed that the sixth figure had not only a reflection but also a rotation symmetry and she added that the statement "rotation was not applied" is explicitly indicated in the explanation. PT15 went on commenting and expressed that there was a contradiction in the written explanation because of the statement "rotation was not applied". This conflict was resolved when PT14 made an explanation as follows: "We were not able to recognize the rotation symmetry while writing our explanation for this figure".

PT7 & PT8 selected the seventh explanation written by PT3 & PT4 and read it loudly but they could not specify to which figure this explanation was belonged to. PT9 pointed to the third figure given in Table 1. PT13 stated that "if there is also a reflection, that figure (i.e, the seventh figure in Table 2) can also match with the explanation" (seventh explanation). Briefly, the participants vacillated between the third and the seventh figures. Indeed, the seventh explanation was not precise and it was true for both figures. Then, the researcher asked PT3 & PT4, those who made an imprecise explanation, to explain the difference between the third and the seventh figure. They replied as "we recognized reflection just now!" The conflict was resolved after PT3 & PT4 verbally indicated that they forgot adding 'reflection' to their explanation.

PT9 & PT10 read aloud the third written explanation they selected from the bag. PT9 interpreted that this explanation might refer to 180 degrees not to 90 degrees rotation since it was expressed that the rotation was applied twice. P11 stated that there cannot be such a figure that has the features indicated in the written explanation. PT7 stated that they wrote that explanation for the third figure. The researcher asked PT7 & PT8 to identify how many rotation symmetries their figure has. They did not give any response to the researcher's question. Meanwhile, some other participants in the classroom stated that the angle of rotation is 120 degrees. PT9 & PT10 had difficulty matching this explanation with relevant figures because of the incorrect information written by PT7 & PT8.

PT11 & PT12 read aloud the fifth written explanation they selected from the bag. There were a few mismatched figures left on the smart board. They indicated in a very short time that this explanation belonged to the fifth figure. In the meantime, all participants cried out as "Home!" to signify the figure. PT3 & PT4 read aloud the second written explanation they selected from the bag. As time proceeded, the participants became impatient and some of them cried out the answer as "Stairs!" without giving PT3 & PT4 the opportunity to state their own answer. PT5 & PT6 read aloud the fourth written explanation they selected from the bag. At that moment, there were only two figures on the board that remained unmatched. They pointed to the fourth figure as a match to their explanation. PT13 & PT14 read the explanation they selected from the bag. When they read the statement "no transformation was applied" and as it was the last figure on the board, they responded that the answer is the first figure.

Briefly, since the sixth and seventh explanations lacked necessary transformations to identify the geometric figures, and the third explanation included errors in it, the participants had difficulty in matching figures with explanations for three out of eight figures.

DISCUSSION AND CONCLUSION

The purpose of this study was to explore prospective middle school mathematics teachers' use of parameters in explaining geometric transformations. Through this purpose, participants were presented finite figures constructed by using geometric transformations and they were expected to explain these finite figures by using geometric transformations. The findings of the study showed that none of the eight pairs provided totally correct explanations for finite figures. Actually, none of the pairs used center of the rotations while explaining finite figures. Besides, only four pairs included the angle of rotation in their explanations. Even, one of the pairs held incorrect ideas about identification of the angle of rotation. For the figures that include both rotation and reflection, three pairs identified only one of these transformations, namely either rotation or reflection. Those who identified reflections could also identify their parameters, namely the reflection lines. As a result, it was found that participants could not use the parameters of geometric transformations adequately while explaining the finite figures constructed by using geometric transformations. This result is consistent with the findings of earlier studies conducted with students and prospective teachers. For instance, Hollebrands, (2004) found out that high school students explained a given rotation by using the angle of rotation explicitly while the center of the rotation seemed to be implicit. Similarly, Thaqi and others (2011) found out that prospective elementary teachers mentioned only the angle of rotation but not the center of rotation when explaining the rotation. Moreover, Harper (2003) presented a figure and its rotated image to four prospective elementary teachers and she found that none of the participants could explain the transformation by using only one rotation. The prospective elementary teachers tried to explain the given rotation by a combination of simple steps without identifying any specific center of rotation and angle of rotation. Similarly, when a figure and its reflected image were presented to prospective elementary teachers, they had difficulty in finding a single reflection line (Harper, 2003).

Participants' difficulty in identifying geometric transformations with their all parameters can be explained by their difficulty in understanding relationships among the parameter(s), a pre-image point and the corresponding image point in geometric transformations. Hollebrands (2004) explained that while solving geometric transformations tasks, students did not carefully consider the relationships among a pre-image, a parameter(s), and an image. In particular, she pointed out that when solving rotation tasks, students did not consider the property that corresponding pre-image and image points are equidistant from the center of rotation. Similarly, Ramful and others (2015) reported that learners were not able to carefully consider that reflection line must be the perpendicular bisector of the segments formed by joining corresponding pre-image and image points.

Parameters and their effects on the transformations are important in understanding geometric transformations. Actually, they are parts of the definitions of geometric transformations. Therefore, the mathematics (e.g., Analytic Geometry) and mathematics education courses (e.g., Methods of Teaching Mathematics) in teacher education programs should help prospective mathematics teachers increase their knowledge and understanding of geometric transformations. Moreover, mathematics teacher educators should design new elective courses that cover all aspects of geometric transformations and that help to teach geometric transformations to their students robustly. Prospective teachers' experiences with such compulsory and elective courses are significant for several reasons. First of all, since they are expected to start their teaching profession in the near future, they must have complete understanding of geometric transformations before entering into the classrooms. Besides, if they have partial understanding of geometric transformations, it is more likely that their students may also have partial understanding of geometric transformations. Even worse, the prospective teachers may convey their difficulties, errors, or misconceptions about geometric transformations to their own students in the course of classroom instruction. Therefore, mathematics educators should design the content of methods of mathematics teaching courses in a way that meets the needs of prospective teachers and help prospective teachers remedy their difficulties, errors, and misconceptions about geometric transformations before they graduate from their programs.

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